

REMLING'S THEOREM ON CANONICAL SYSTEMS

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ABSTRACT. We prove the Remling's Theorem on canonical systems and discuss the connection between Jacobi and Schrödinger equation and canonical systems.

Keywords: Canonical systems, Weyl-m functions, absolutely continuous spectrum, reflectionless Hamiltonians.

1. INTRODUCTION

This paper deals with the canonical system of the following form

$$(1.1) \quad Ju'(x) = zH(x)u(x).$$

Here $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and $H(x)$ is a 2×2 positive semidefinite matrix whose entries are locally integrable and that there is no non-empty open interval I so that $H = 0$ *a.e.* on I . The complex number $z \in \mathbb{C}$ involved in 1.1 is a spectral parameter. For fixed $z \in \mathbb{C}$, a function $u(., z) : [0, N] \rightarrow \mathbb{C}^2$ is called a solution if u is absolutely continuous and satisfies 1.1. Consider the Hilbert space

$$L^2(H, \mathbb{R}_+) = \left\{ f(x) = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix} : \|f\| < \infty \right\}$$

with an inner product $\langle f, g \rangle = \int_0^\infty f(x)^* H(x) g(x) dx$. Such canonical systems 1.1 on $L^2(H, \mathbb{R}_+)$ have been studied by Hassi, De Snoo, Winkler, and Remling in the papers [10], [13], [16], [9] in various context. The Jacobi and Schrödinger equations can be written into canonical systems with appropriate choice of $H(x)$. In addition, the canonical systems are closely connected with the theory of de Branges spaces and the inverse spectral theory of one dimensional Schrödinger equations, see [13].

Recently, in the spectral theory of Jacobi and Schrödinger operators, the Remling's theorem, published in the *Annals of Math* in 2011 (see [14]), has been one of the most popular results. It has revealed some new fundamental properties of absolutely continuous spectrum of Jacobi and Schrödinger operators that changed the perspective of many mathematicians about the absolutely continuous spectrum. In this paper we will prove the Remling's Theorem on canonical systems.

This paper has been organized as follows: In section 2, we discuss the Weyl theory of canonical systems following the analogous treatment of Weyl theory of Jacobi and Schrödinger equations. In section 3 we discuss the basic definitions and space of Hamiltonians in order to state the main theorem. In section 4 we prove our main theorem using the similar techniques from [14], more specifically we prove the Breimesser-Pearson theorem on canonical systems which is in fact the foundation

for the proof of the Remling's theorem. Finally we show the connection between Jacobi and Schrödinger equations with canonical systems in section 5.

2. WEYL THEORY

Let u_α, v_α be the solution of 1.1 with the initial values

$$u_\alpha(0, z) = \begin{pmatrix} \cos \alpha \\ -\sin \alpha \end{pmatrix} \quad \text{and} \quad v_\alpha(0, z) = \begin{pmatrix} \sin \alpha \\ \cos \alpha \end{pmatrix}, \quad \alpha \in (0, \pi].$$

For $z \in \mathbb{C}^+$, want to define $m_\alpha(z) \in \mathbb{C}$ as the unique coefficient for which

$$f_\alpha = u_\alpha + m_\alpha(z)v_\alpha \in L^2(H, \mathbb{R}_+).$$

Consider a compact interval $[0, N]$ and let $z \in \mathbb{C}^+$, define the unique coefficient $m_N^\beta(z)$ as follows, $f(x, z) = u(x, z) + m_N^\beta(z)v(x, z)$ satisfying

$$f_1(N, z) \sin \beta + f_2(N, z) \cos \beta = 0.$$

Clearly this is well defined because $u(x, z)$ does not satisfies the boundary condition at N . Otherwise $z \in \mathbb{C}^+$ will be an eigen value for some self-adjoint relation of the system 1.1 as explained in [1]. From the boundary condition $f_1(N, z) \sin \beta + f_2(N, z) \cos \beta = 0$ at N we get

$$m_N^\beta(z) = -\frac{u_1(N, z) \sin \beta + u_2(N, z) \cos \beta}{v_1(N, z) \sin \beta + v_2(N, z) \cos \beta}.$$

As z, N, β varies $m_N^\beta(z)$ becomes a funtion of these arguments, and since u_1, u_2, v_1, v_2 are entire function of z it follows that $m_N^\beta(z)$ is meromorphic function of z .

Let $C_N(z) = \{m_N^\beta(z) : 0 \leq \beta < \pi\}$

Here

$$m_N^\beta(z) = -\frac{u_1 t + u_2}{v_1 t + v_2}, \quad t = \tan \beta, \quad t \in \mathbb{R} \cup \{\infty\}.$$

This is a fractional linear transformation. As a function of $t \in \mathbb{R}$ it maps real line to a circle. So for fixed $z \in \mathbb{C}^+$, $C_N(z)$ is a circle. Hence for any complex number $m \in \mathbb{C}$

$$m \in C_N(z) \Leftrightarrow \operatorname{Im} \frac{u_2 + mv_2}{u_1 + mv_1} = 0$$

From this identity the equation of the circle $C_N(z)$ is given by $|m - c|^2 = r^2$ where

$$(2.1) \quad c = \frac{W_N(u, \bar{v})}{W_N(\bar{v}, v)}, \quad r = \frac{1}{|W_N(\bar{v}, v)|}$$

Now suppose $f(x, z) = u(x, z) + m_N^\beta(z)v(x, z)$, then $m = m_N^\beta$ is an interior of C_N if and only if

$$(2.2) \quad |m - c|^2 < r^2 \Leftrightarrow \frac{W_N(\bar{f}, f)}{W_N(\bar{v}, v)} < 0$$

Let us write $\tau y = zy$ if and only if $Jy' = zH(x)y$. Suppose f and g are the solutions of 1.1 then we have the following identity, called the *Green's Identity*.

$$(2.3) \quad \int_0^N (f^* H(x) \tau g - (\tau f)^* H(x) g(x)) dx = W_0(\bar{f}, g) - W_N(\bar{f}, g)$$

Using the Green's identity we have,

$$(2.4) \quad W_N(\bar{f}, f) = 2i \operatorname{Im} m(z) - 2i \operatorname{Im} z \int_0^N f^*(x)H(x)f(x)dx.$$

$$W_N(\bar{v}, v) = -2i \operatorname{Im} z \int_0^N v^*(x)H(x)v(x)dx.$$

$$\frac{W_N(\bar{f}, f)}{W_N(\bar{v}, v)} = \frac{-\operatorname{Im} m(z) + \operatorname{Im} z \int_0^N f^*(x)H(x)f(x)dx}{\operatorname{Im} z \int_0^N v^*(x)H(x)v(x)dx}.$$

Hence from 2.2 we see that $\frac{W_N(\bar{f}, f)}{W_N(\bar{v}, v)} < 0$ if and only if

$$\int_0^N f^*(x)H(x)f(x)dx < \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}.$$

Thus it follows that m is an interior of C_N if and only if

$$(2.5) \quad \int_0^N f^*(x)H(x)f(x)dx < \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}$$

and $m \in C_N(z)$ if and only if

$$(2.6) \quad \int_0^N f^*(x)H(x)f(x)dx = \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}.$$

For $z \in \mathbb{C}^+$, the radius of the circle $C_N(z)$ is given by

$$(2.7) \quad r_N(z) = \frac{1}{|W_N(\bar{v}, v)|} = \frac{1}{2 \operatorname{Im} z \int_0^N v^*(x)H(x)v(x)dx}.$$

Now let $0 < N_1 < N_2 < \infty$. Then if m is inside or on C_{N_2}

$$\int_0^{N_1} f^*(x, z)H(x)f(x, z)dx < \int_0^{N_2} f^*(x, z)H(x)f(x, z)dx \leq \frac{\operatorname{Im} m}{\operatorname{Im} z}$$

and therefore m is inside C_{N_1} . Let us denote the interior of $C_N(z)$ by $\operatorname{Int} C_N(z)$ and suppose $D_N(z) = C_N(z) \cup \operatorname{Int} C_N(z)$. Then

$$m \in D_N(z) \Leftrightarrow \int_0^N f^*(x)H(x)f(x)dx \leq \frac{\operatorname{Im} m(z)}{\operatorname{Im} z}.$$

These are called the Wyle Disks. These Wyle Disks are nested. That is $D_{N+\epsilon}(z) \subset D_N(z)$ for any $\epsilon > 0$. From 2.7 we see that $r_N(z) > 0$, and $r_N(z)$ decreases as $N \rightarrow \infty$. So $\lim_{N \rightarrow \infty} r_N(z)$ exists and

$$\lim_{N \rightarrow \infty} r_N(z) = 0 \Leftrightarrow v \notin L^2(H, \mathbb{R}_+).$$

Thus for a given $z \in \mathbb{C}^+$ as $N \rightarrow \infty$ the circles $C_N(z)$ converges either to a circle $C_\infty(z)$ or to a point $m_\infty(z)$. If $C_N(z)$ converges to a circle, then its radius $r_\infty = \lim r_N$ is positive and 2.7 implies that $v \in L^2(H, \mathbb{R}_+)$. If \tilde{m}_∞ is any point on $C_\infty(z)$ then \tilde{m}_∞ is inside any $C_N(z)$ for $N > 0$. Hence

$$\int_0^N (u + \tilde{m}_\infty v)^* H(u + \tilde{m}_\infty v) < \frac{\operatorname{Im} \tilde{m}_\infty}{\operatorname{Im} z}$$

and letting $N \rightarrow \infty$ one sees that $f(x, z) = u + \tilde{m}_\infty v \in L^2(H, \mathbb{R}_+)$. The same argument holds if \tilde{m}_∞ reduces to a point m_∞ . Therefore, if $\text{Im } z \neq 0$, there always exists a solution of 1.1 of class $\in L^2(H, \mathbb{R}_+)$. In the case $C_N(z) \rightarrow C_\infty(z)$ all solutions are in $L^2(H, \mathbb{R}_+)$ for $\text{Im } z \neq 0$ and this identifies the limit-circle case with the existence of the circle $C_\infty(z)$. Correspondingly, the limit-point case is identified with the existence of the point $m_\infty(z)$. In this case $C_N(z) \rightarrow m_\infty$ there results $\lim r_N = 0$ and 2.7 implies that v is not of class $L^2(H, \mathbb{R}_+)$. Therefore in this situation there is only one linearly independent solution of class $L^2(H, \mathbb{R}_+)$. In the limit circle case $m \in C_N$ if and only 2.6 holds. Since $f(x, z) = u(x, z) + mv(x, z)$, it follows that m is on C_∞ if and only if

$$(2.8) \quad \int_0^\infty f(x, z)^* H f(x, z) dx = \frac{\text{Im } m(z)}{\text{Im } z}.$$

From 2.4, it follows that m is on the limit circle if and only if $\lim_{N \rightarrow \infty} W_N(\bar{f}, f) = 0$. From the above discussion we proved

Theorem 2.1. [6]

- (1) *The limit-point case ($r_\infty = 0$) implies that 1.1 has precisely one $L^2(H, \mathbb{R}_+)$ solution.*
- (2) *The limit-circle case ($r_\infty > 0$) implies all solutions of 1.1 are in $L^2(H, \mathbb{R}_+)$.*

The identity 2.6 shows that $m_N^\beta(z)$ are holomorphic functions mapping upper-half plane to itself. The poles and zeros of these functions lie on the real line and are simple. In the limit-point case, the limit $m_\infty(z)$ is a holomorphic function mapping upper-half plane to itself. In limit-circle case, each circle $C_N(z)$ is traced by points $m = m_N^\beta(z)$ as β ranges over $0 \leq \beta < \pi$ for fixed N and z . Let z_0 be fixed, $\text{Im } z_0 > 0$. A point $\tilde{m}_\infty(z_0)$ on the circle $C_\infty(z_0)$ is the limit point of a sequence $m_{N_j}^{\beta_j}(z)$ with $N_j \rightarrow \infty$ as $j \rightarrow \infty$.

In [1] we showed that $\text{tr } H \equiv 1$ implies the limit-point case. We would like to consider 1.1 to have limit-point case. If the Hamiltonian H in a canonical system does not have trace norm 1, then we can always pass the equivalent canonical system with the Hamiltonian H having trace norm 1. More precisely we use the change of variable

$$(2.9) \quad t(x) = \int_0^x \text{tr } H(s) ds.$$

Let $x(t)$ be the inverse function and define the new Hamiltonian $\tilde{H}(t) = \frac{1}{\text{tr } H(x)} H(x(t))$ so that $\text{tr } \tilde{H}(t) \equiv 1$. Let $u(x, z)$ be the solution of the original system

$$Ju' = zHu$$

and put $\tilde{u}(t, z) = u(x(t), z)$. Then $\tilde{u}(t, z)$ solves the new equation

$$J\tilde{u}' = z\tilde{H}\tilde{u}.$$

Their corresponding Weyl-m functions on a compact interval $[0, N]$ are the same up to the change of the point of boundary condition, ie $\tilde{m}_N^\beta(z) = m_{x(N)}^\beta(z)$. From now onward we will consider a canonical system with $\text{tr } H \equiv 1$. This will reduce our system to have limit-point case.

3. TOPOLOGIES ON THE SPACE OF HAMILTONIAN.

We need to consider the space of Hamiltonians and a suitable topology on it so that the space is compact. With the topology we have, we want to work with the basic object like m functions of the canonical system. Let $M(\mathbb{R})$ denotes the set of Borel measures on \mathbb{R} . Consider the space

$$\mathcal{V}_{2 \times 2} = \{\mu \in M(\mathbb{R})^{2 \times 2} : d\mu = H(x)dx, H(x) \geq 0, \text{tr } H(x) \equiv 1, H(x) \in L^1_{\text{loc}}\}.$$

We would like to define a metric on $\mathcal{V}_{2 \times 2}$. We will follow the same procedure as in 4.1. Let $C_c(\mathbb{R})$ denotes the space of all continuous functions on \mathbb{R} with compact support, the continuous functions vanishing outside of a bounded interval. This space $C_c(\mathbb{R})$ is complete with respect to the $\|\cdot\|_\infty$ norm. Pick a countable dense subset $\{f_n : n \in \mathbb{N}\} \subset C_c(\mathbb{R})$, the continuous functions of compact support. Let

$$\rho_n(\mu, \nu) = \sum_{1 \leq i, j \leq 2} \left| \int f_n d(\mu_{ij} - \nu_{ij})(x) \right|.$$

Then define a metric d on $\mathcal{V}_{2 \times 2}$. as

$$d(\mu, \nu) = \sum_{n=1}^{\infty} 2^{-n} \frac{\rho_n(\mu, \nu)}{1 + \rho_n(\mu, \nu)}.$$

Clearly d is a metric on $\mathcal{V}_{2 \times 2}$. Moreover, $(\mathcal{V}_{2 \times 2}, d)$ is a compact metric space.

We can now consider canonical system with measures as Hamiltonian.

$$(3.1) \quad Ju' = z\mu u, \quad \mu \in \mathcal{V}_{2 \times 2}.$$

If $I \subset \mathbb{R}$ is an compact interval and $B(I)$ denotes the space of all complex valued bounded Borel measurable functions on I . Then $B(I)$ is complete with respect to the metric given by $\rho(f, g) = \|f - g\|_u$ where the norm on $B(I)$ is $\|f\|_u = \sup_{x \in I} |f(x)|$.

Let $B(I)^2 = \left\{ f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} : f_1, f_2 \in B(I) \right\}$. Clearly the space $B(I)^2$ is complete with respect to the metric given by $\rho(f, g) = \|f - g\|_u$ where $\|f\|_u = \sup_{x \in I} |f_1(x)| + \sup_{x \in I} |f_2(x)|$. Let $f \in B(I)^2$, we call f a solution to the equation 3.1 if and only if

$$J(u(x) - u(a+)) = z \int_{(a, x)} d\mu(t) u(t) \quad \text{if } x \geq a \geq 0$$

and

$$J(u(x) - u(a-)) = -z \int_{(x, a)} d\mu(t) u(t) \quad \text{if } x \leq a \leq 0.$$

In order to show the existence of a solution of the equation 3.1, define a map on $B(I)^2$ by

$$Tu(x) = u(0) - zJ \int_0^x \mu(t) u(t)$$

and show that T is a contraction mapping.

$$\begin{aligned}
\|Tu - Tv\|_u &= \sup_{x \in I} \left| z \int_0^x (u_1 - v_1) d\mu_{11} + (u_2 - v_2) d\mu_{12} \right| \\
&\quad + \sup_{x \in I} \left| z \int_0^x (u_1 - v_1) d\mu_{21} + (u_2 - v_2) d\mu_{22} \right| \\
&\leq \sup_{x \in I} \left[|z| \int_0^x |u_1 - v_1| d\mu_{11} + |z| \int_0^x |u_2 - v_2| d\mu_{12} \right. \\
&\quad \left. + \sup_{x \in I} |z| \int_0^x |u_1 - v_1| d\mu_{21} + |z| \int_0^x |u_2 - v_2| d\mu_{22} \right] \\
&\leq \frac{c}{2} \sup_{x \in I} \left[\sup_{t \in [0, x]} |u_1 - v_1| + \sup_{t \in [0, x]} |u_2 - v_2| \right] \\
&\quad + \frac{c}{2} \sup_{t \in [0, x]} \left[\sup_{t \in [0, x]} |u_1 - v_1| + \sup_{t \in [0, x]} |u_2 - v_2| \right] \\
&\leq \frac{c}{2} \left[2 \sup_{x \in I} |u_1 - v_1| + 2 \sup_{x \in I} |u_2 - v_2| \right] \\
&= c \|u - v\|_u.
\end{aligned}$$

This shows that T is a contraction mapping, so it has a unique fixed point say $u(x)$ in $B(I)^2$ such that $Tu(x) = u(x)$. So there is a solution in $B(I)^2$ that satisfy $u(x) = u(0) - zJ \int_0^x \mu(t)u(t)$. As already seen that $\text{tr } H(x) \equiv 1$ implies the limit point at the both end points.

This means that for $z \in \mathbb{C}^+$ there exists (unique up to a factor) solutions $f_{\pm}(x, z) = u(x, z) \pm m_{\pm}(z)v(x, z)$ of 3.1 such that $f_- \in L^2(H, \mathbb{R}_-)$, $f_+ \in L^2(H, \mathbb{R}_+)$ where $u(x, z)$ and $v(x, z)$ are any two linearly independent solutions of 3.1. Let $x \in \mathbb{R}$, and $\alpha = 0$, the Dirichlet boundary condition at x that is $u_1(x, z) = v_2(x, z) = 0, v_1(x, z) = u_2(x, z) = 1$, the Titchmarsh-Weyl m -functions of the system 3.1 are alternately defined as $m_{\pm}(x, z) = \pm \frac{f_{\pm 2}(x, z)}{f_{\pm 1}(x, z)}$. Recall that $m_{\pm}(x, z)$ are Herglotz functions. So the boundary value of these m functions are defined by $m_{\pm}(x, t) \equiv \lim_{y \rightarrow 0} m_{\pm}(x, t + iy)$.

Definition 3.1. Let $A \subset \mathbb{R}$ be a Borel set. We call a Hamiltonian $\mu \in \mathcal{V}_{2 \times 2}$ reflectionless on A if

$$(3.2) \quad m_+(x, t) = -\overline{m_-(x, t)}$$

for almost every $t \in A$ and for some $x \in \mathbb{R}$.

The set of reflectionless hamiltonian on A is denoted by $\mathcal{R}(A)$. Notice that the equation 3.2 is independent of the choice of boundary condition and the choice of a point. Suppose 3.2 is true for a boundary condition α at 0. Let $m_+^{\alpha}(z)$ be the unique coefficient such that $f(x, z) = u_{\alpha}(x, z) + m_+^{\alpha}(z)v_{\alpha}(x, z) \in L^2(H, \mathbb{R}_+)$.

Suppose $T_{\alpha}(x, z) = \begin{pmatrix} u_{\alpha 1}(x, z) & v_{\alpha 1}(x, z) \\ u_{\alpha 2}(x, z) & v_{\alpha 2}(x, z) \end{pmatrix}$ with $T_{\alpha}(0, z) = \begin{pmatrix} \sin \alpha & \cos \alpha \\ -\cos \alpha & \sin \alpha \end{pmatrix}$ and

$T_{\beta}(x, z) = \begin{pmatrix} u_{\beta 1}(x, z) & v_{\beta 1}(x, z) \\ u_{\beta 2}(x, z) & v_{\beta 2}(x, z) \end{pmatrix}$ with $T_{\beta}(0, z) = \begin{pmatrix} \sin \beta & \cos \beta \\ -\cos \beta & \sin \beta \end{pmatrix}$. Then

$T_{\alpha}(x, z) = T_{\beta}(x, z) \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix}$, where $\gamma = \beta - \alpha$.

Here $m_+^\alpha(z) \in \mathbb{C}$ is a unique number such that

$$\begin{aligned} T_\alpha(x, z) \begin{pmatrix} 1 \\ m_+^\alpha(z) \end{pmatrix} &\in L^2(H, \mathbb{R}_+). \\ \Rightarrow T_\beta(x, z) \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} 1 \\ m_+^\alpha(z) \end{pmatrix} &\in L^2(H, \mathbb{R}_+). \\ \Rightarrow T_\beta(x, z) \begin{pmatrix} \cos \gamma + m_+^\alpha(z) \sin \gamma \\ -\sin \gamma + m_+^\alpha(z) \cos \gamma \end{pmatrix} &\in L^2(H, \mathbb{R}_+). \\ \Rightarrow (\cos \gamma + m_+^\alpha(z) \sin \gamma) T_\beta(x, z) \begin{pmatrix} 1 \\ \frac{-\sin \gamma + m_+^\alpha(z) \cos \gamma}{\cos \gamma + m_+^\alpha(z) \sin \gamma} \end{pmatrix} &\in L^2(H, \mathbb{R}_+). \end{aligned}$$

Since $m_+^\beta(z)$ be the unique coefficient such that $T_\beta(x, z) \begin{pmatrix} 1 \\ m_+^\beta(z) \end{pmatrix} \in L^2(H, \mathbb{R}_+)$ we must have,

$$m_+^\beta(z) = \frac{-\sin \gamma + m_+^\alpha(z) \cos \gamma}{\cos \gamma + m_+^\alpha(z) \sin \gamma} = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix} m_+^\alpha(z).$$

On the other hand, exactly in the same way,

$$\begin{aligned} T_\alpha(x, z) \begin{pmatrix} 1 \\ -m_-^\alpha(z) \end{pmatrix} &\in L^2(H, \mathbb{R}_-). \\ \Rightarrow T_\beta(x, z) \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} \begin{pmatrix} 1 \\ -m_-^\alpha(z) \end{pmatrix} &\in L^2(H, \mathbb{R}_-), \text{ where } \gamma = \beta - \alpha. \\ \Rightarrow T_\beta(x, z) \begin{pmatrix} \cos \gamma - m_-^\alpha(z) \sin \gamma \\ -\sin \gamma - m_-^\alpha(z) \cos \gamma \end{pmatrix} &\in L^2(H, (-\infty, 0]) \\ \Rightarrow (\cos \gamma - m_-^\alpha(z) \sin \gamma) T_\beta(x, z) \begin{pmatrix} 1 \\ \frac{-\sin \gamma - m_-^\alpha(z) \cos \gamma}{\cos \gamma - m_-^\alpha(z) \sin \gamma} \end{pmatrix} &\in L^2(H, \mathbb{R}_-) \\ \Rightarrow -m_-^\beta(z) = \frac{-\sin \gamma - m_-^\alpha(z) \cos \gamma}{\cos \gamma - m_-^\alpha(z) \sin \gamma} = \begin{pmatrix} \cos \gamma & \sin \gamma \\ -\sin \gamma & \cos \gamma \end{pmatrix} m_-^\alpha(z). \end{aligned}$$

Let $P_+(0, z) = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix}$ and $P_-(0, z) = \begin{pmatrix} \cos \gamma & -\sin \gamma \\ \sin \gamma & \cos \gamma \end{pmatrix}$, so that

$m_-^\beta(z) = P_-(0, z)m_-^\alpha(z)$, $m_+^\beta(z) = P_+(0, z)m_+^\alpha(z)$ and $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} P_+(0, z) = P_-(0, z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. By simple calculation we can see that

$$m_+^\beta(t) = \overline{-m_-^\beta(t)}.$$

Similarly, equation 3.2 is independent of the choice of the point. Suppose

$T_0(x, z) = \begin{pmatrix} u_1(x, z) & v_1(x, z) \\ u_2(x, z) & v_2(x, z) \end{pmatrix}$ be solutions with the boundary conditions at 0.

Then $T_0(x, z) = T_a(x, z) \begin{pmatrix} u_1(a, z) & v_1(a, z) \\ u_2(a, z) & v_2(a, z) \end{pmatrix}$. Suppose $m_\pm(0, z) \in \mathbb{C}$ be the unique coefficients such that $f_\pm(x, z) = u(x, z) \pm m_\pm(0, z)v(x, z) \in L^2(H, \mathbb{R}_\pm)$.

In another way, $T_0(x, z) \begin{pmatrix} 1 \\ \pm m_\pm(0, z) \end{pmatrix} \in L^2(H, \mathbb{R}_\pm)$.

$$\Rightarrow T_a(x, z) \begin{pmatrix} u_1(a, z) & v_1(a, z) \\ u_2(a, z) & v_2(a, z) \end{pmatrix} \begin{pmatrix} 1 \\ \pm m_\pm(0, z) \end{pmatrix} \in L^2(H, \mathbb{R}_\pm).$$

$\Rightarrow m_{\pm}(a, z) = \frac{u_2(a, z) \pm m_{\pm}(0, z) v_2(a, z)}{u_1(a, z) \pm m_{\pm}(0, z) v_1(a, z)} = \begin{pmatrix} v_2(a, z) & \pm u_2(a, z) \\ \pm v_1(a, z) & u_1(a, z) \end{pmatrix} m_{\pm}(0, z).$

Let $T_{\pm}(z) = \begin{pmatrix} v_2(a, z) & \pm u_2(a, z) \\ \pm v_1(a, z) & u_1(a, z) \end{pmatrix}$, then $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} T_{+}(z) = T_{-}(z) \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

By calculation we see that

$$m_{+}(a, t) = -\overline{m_{-}(a, t)}$$

As already mentioned, the Weyl m-functions $m(x, \cdot)$ are Herglotz functions and by the Herglotz representation theorem they have unique integral representation of the form,

$$m(x, z) = a + bz + \int_{\mathbb{R}} \left(\frac{1}{t - z} - \frac{t}{t^2 + 1} \right) d\nu(t), \quad z \in \mathbb{C}^{+}$$

for some positive Borel measure ν on \mathbb{R} with $\int \frac{1}{t^2 + 1} d\nu < \infty$ and numbers $a \in \mathbb{R}$, $b \geq 0$. We call the measure ν in above integral representation of m as *spectral measure* of the system 1.1.

Recall that a Borel measure ρ on \mathbb{R} is called *absolutely continuous* if $\rho(B) = 0$ for all Borel sets $B \subset \mathbb{R}$ of Lebesgue measure zero. By the Radon-Nikodym Theorem, ρ is absolutely continuous if and only if $d\rho = f(t)dt$ for some density $f \in L^1_{loc}(\mathbb{R})$, $f \geq 0$. If ν is supported by a Lebesgue null set that is, there exists a Borel set $B \subset \mathbb{R}$ with $|B| = \rho(B^c) = 0$, then we say that ρ is *singular*. By Lebesgue's decomposition theorem, any Borel measure ρ on \mathbb{R} can uniquely decomposed into absolutely continuous and singular parts:

$$\rho = \rho_{ac} + \rho_s.$$

The *essential support* Σ_e of a Borel measure ρ on \mathbb{R} is the complement of a largest open set $U \subset \mathbb{R}$ such that $\rho(U) = 0$.

Let μ be a measure on \mathbb{R} . The shift by x of the measure μ , denoted by $S_x \mu$, is defined by

$$\int_{\mathbb{R}} f(t) d(S_x \mu) = \int_{\mathbb{R}} f(t - x) d\mu(t).$$

If $\mu \in \mathcal{V}_{2 \times 2}$ is such that $d\mu = H(t)dt$ is a locally integrable Hamiltonian H , then this reduces to the shift map $(S_x H)(t) = H(x + t)$.

Definition 3.2. The ω limit set of the Hamiltonian $\mu \in \mathcal{V}_{2 \times 2}$ under the shift map is defined as,

$$\omega(\mu) = \{ \nu \in \mathcal{V}_{2 \times 2} : \text{there exist } x_n \rightarrow \infty \text{ so that } d(S_{x_n} \mu, \nu) \rightarrow 0 \}.$$

Then as in [15] we can see that $\omega(\mu) \subset \mathcal{V}_{2 \times 2}$ is compact, non-empty and S is a homeomorphism on $\omega(\mu)$. Moreover, $\omega(\mu)$ is connected.

4. MAIN THEOREM AND ITS PROOF

We are now ready to state the Remling's theorem for Canonical System on \mathbb{R}_+ .

Theorem 4.1. Let $\mu \in \mathcal{V}_{2 \times 2}$ be a (half line) Hamiltonian, and let Σ_{ac} be the essential support of the absolutely continuously part of the spectral measure. Then

$$\omega(\mu) \subset \mathcal{R}(\Sigma_{ac}).$$

In order to prove this theorem we approach the similar way as in [15]. Let $\mu \in \mathcal{V}_{2 \times 2}$ is a whole line Hamiltonian. We write μ_{\pm} for the restrictions of μ to \mathbb{R}_{\pm} . Denote the set of restrictions by $\mathcal{V}_{\pm} = \{\mu_{\pm} : \mu \in \mathcal{V}_{2 \times 2}\}$.

Let \mathbb{H} denote the set of all Herglotz functions, that is $\mathbb{H} = \{F : \mathbb{C}^+ \rightarrow \mathbb{C}^+ : F \text{ is holomorphic}\}$. So $\{M_{\pm} = m_{\pm}^{\mu}(0, z)\} \subset \mathbb{H}$. First lets prove the following lemma.

Lemma 4.2. *The maps $\mathcal{V}_{\pm} \mapsto \mathbb{H}$, $\mu_{\pm} \mapsto M_{\pm} = m_{\pm}^{\mu}(0, z)$ are homeomorphism onto their images.*

Proof. We have $\mu_+ = H_+(x)dx$. By Theorem 1 in [3] for every canonical system with Hamiltonian H_+ there is unique $m_+(0, z)$. Conversely for every $m_+ \in \mathbb{H}$ there exists a unique Hamiltonian H_+ on \mathbb{R}_+ such that m_+ is a Wyle coefficient of the canonical system corresponding to H_+ . So $\mu_+ \mapsto M_+$ is one-to-one. Next we show that the map is homeomorphism. Consider the canonical system 1.1 on \mathbb{R}_+ . Suppose $\mu_n \rightarrow \mu$ in \mathcal{V}_+^C . That is $H_n(x)dx \rightarrow H(x)dx$ for some Hamiltonian $H_n(x), H(x)$. Let u_n be the solution of Canonical System with Hamiltonian $H_n(x)$. Let K be a compact subset of \mathbb{C}^+ contained in a ball $B(0, R)$. Suppose a subinterval $[0, \eta]$ be such that $\eta = \frac{1}{8R}$. We claim that u_n has convergent subsequence on $[0, \eta]$. Define the operators $T_n : C[0, \eta] \rightarrow C[0, \eta]$ by

$$T_n u(x) = -zJ \int_0^x H_n(t)u(t)dt.$$

Since

$$\begin{aligned} \|T_n\| &= \sup_{\|u\|_{\infty}=1} \left\| -zJ \int_0^x H_n(t)u(t)dt \right\| \\ &\leq |z| \|u\|_{\infty} \int_0^x |H_n(t)| dt \\ &\leq R4\eta = R4 \frac{1}{8R} = \frac{1}{2}, \end{aligned}$$

$\|T_n\|$ are uniformly bounded. So the Neumann series $(1 - T_n)^{-1} = \sum_{k=0}^{\infty} T_n^k$ is convergent. Here $u_n(x) = (1 - T_n)^{-1}(u_0)$, $u_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

$\|u_n\| \leq \|(1 - T_n)^{-1}\| \|u_0\| = \|(1 - T_n)^{-1}\| \leq \sum_{k=0}^{\infty} \|T_n\|^k = \sum_{k=0}^{\infty} \left(\frac{1}{2}\right)^k = 2$. So $\{u_n(x) = n \in \mathbb{N}\}$ is uniformly bounded in n on $[0, \eta]$ and locally uniformly in z . Similar argument shows that u_n remains bounded on $[\eta, \eta + p]$ so that u_n are eventually bounded uniformly on $[0, N]$. Moreover, u_n are equicontinuous. Let $\epsilon > 0$ be given. Since u_n are solutions for the system 1.1 we have,

$$u_n(x) - u_n(x_0) = -zJ \int_{x_0}^x H_n(t)u_n(t)dt.$$

$$\begin{aligned}
\|u_n(x) - u_n(x_0)\| &\leq |z| \|u_n\| \int_{x_0}^x |H_n(t)| dt \\
&= |z| \|u_n\| 4\eta \|x - x_0\| \\
&\leq R 2.4\eta \|x - x_0\|.
\end{aligned}$$

Let $\delta = \frac{\epsilon}{8R\eta}$ then $\|u_n(x) - u_n(x_0)\| < \epsilon$, if $\|x - x_0\| < \delta$ for all n . By Arzella-Ascoli Theorem $\{u_n\}$ has convergent subsequence say $u_{n_j} \rightarrow u$. We show that u satisfies the Canonical System corresponding to $H(x)$.

$$\begin{aligned}
u_{n_j}(x) - u_{n_j}(0) &= -zJ \int_0^x H_{n_j}(t) u_{n_j}(t) dt \\
&= -zJ \int_0^x H_{n_j}((t)u_{n_j}(t) - u(t)) dt - zJ \int_0^x H_{n_j}(t) u(t) dt.
\end{aligned}$$

Since $\| -zJ \int_0^x H_{n_j}((t)u_{n_j}(t) - u(t)) dt \| \leq |z| \|H_{n_j}\|_{L_1(0,x)} \|u_{n_j} - u\|$,

$\lim_{j \rightarrow \infty} -zJ \int_0^x H_{n_j}((t)u_{n_j}(t) - u(t)) dt = 0$. Hence, taking the limit as $j \rightarrow \infty$ we get, $u(x) - u(0) = \int_0^x H(t)u(t) dt$. So $\mu_n \rightarrow \mu \Rightarrow u_n \rightarrow u \Rightarrow m_+^{\mu_n}(0, z) \rightarrow m_+(0, z)$. This proves the continuity of the map on the interval $[0, N]$. Inverse of a continuous map on compact set is also continuous. Hence the map is homeomorphic. Exactly, the same way $\mu_- \longleftrightarrow M_-$ is also a homeomorphism. \square

4.1. Breimesser-Pearson Theorem on canonical systems. For $z = x + iy \in \mathbb{C}^+$, $\omega_z(S) = \frac{1}{\pi} \int_S \frac{y}{(t-x)^2 + y^2} dt$, denotes the harmonic measure in the upper-half plane. For any $G \in \mathbb{H}$ and $t \in \mathbb{R}$ we define $\omega_{G(t)}(S)$ as the limit

$$\omega_{G(t)}(S) = \lim_{y \rightarrow 0+} \omega_{G(t+iy)}(S).$$

For complete description about the Herglotz functions and harmonic measures, see [15].

Lemma 4.3. [15] *Let $A \subset \mathbb{R}$ be a Borel Set with $|A| < \infty$. Then*

$$\lim_{y \rightarrow 0+} \sup_{F \in \mathcal{H}; S \subset \mathbb{R}} \left| \int_A \omega_{F(t+iy)}(S) dt - \int_A \omega_{F(t)}(S) dt \right| = 0$$

Definition 4.4. *If $F_n, F \in \mathbb{H}$, we say that $F_n \rightarrow F$ in value distribution if*

$$(4.1) \quad \lim_{n \rightarrow \infty} \int_A \omega_{F_n(t)}(S) dt = \int_A \omega_{F(t)}(S) dt$$

for all Borel set $A, S \subset \mathbb{R}, |A| < \infty$.

Notice that if the limit in the value distribution exists, it is unique.

Theorem 4.5. [15] *Suppose $F_n, F \in \mathcal{H}$, and let a_n, a , and ν_n, ν be the associated numbers and measures, respectively, from the integral representation of Herglotz function. Then the following are equivalent:*

- (1) $F_n(z) \rightarrow F(z)$ uniformly on compact subsets of \mathbb{C}^+ ;
- (2) $a_n \rightarrow a$ and $\nu_n \rightarrow \nu$ weak $*$ on $\mathcal{M}(\mathbb{R}_\infty)$, that is,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}_\infty} f(t) d\nu_n(t) = \int_{\mathbb{R}_\infty} f(t) d\nu(t)$$

for all $f \in C(\mathbb{R}_\infty)$;

(3) $F_n \rightarrow F$ in value distribution.

Remling's theorem is in fact a reformulation of Breimesser-Pearson theorem which state as follows

Theorem 4.6. *Consider a half-line Canonical System. Let Σ_{ac} denotes the essential support of absolutely continuous part of Spectral measure then for any $A \subset \Sigma_{ac}$, $|A| < \infty$ and $S \subset \mathbb{R}$, we have*

$$\lim_{N \rightarrow \infty} \left(\int_A \omega_{m-}(N,t)(-S)dt - \int_A \omega_{m+}(N,t)(S)dt \right) = 0.$$

Moreover, the convergence is uniform in S .

We prove this theorem on canonical systems using the same technique as in [15].

The hyperbolic distance of two points $w, z \in \mathbb{C}^+$ is defined as

$$\gamma(w, z) = \frac{|w - z|}{\sqrt{\operatorname{Im} w} \sqrt{\operatorname{Im} z}}.$$

Hyperbolic distance and harmonic measure are intimately related as follows,

$$|\omega_w(S) - \omega_z(S)| \leq \gamma(w, z)$$

for any $z, w \in \mathbb{C}^+$ and any Borel set $S \subset \mathbb{R}$. Moreover, if $F(z) = \alpha(z) + i\omega_z(S)$, $\alpha(z)$ is a harmonic conjugate of $\omega_z(S)$ we have

$$(4.2) \quad |\omega_w(S) - \omega_z(S)| \leq \frac{|\omega_w(S) - \omega_z(S)|}{\sqrt{\omega_w(S)} \sqrt{\omega_z(S)}} \leq \gamma(F(w), F(z)) \leq \gamma(w, z).$$

Lemma 4.7. *Let $u(\cdot, z), v(\cdot, z)$ be the solution of the Canonical System 1.1, subject to the condition $u(0, z) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v(0, z) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Let w be any constant such that $\operatorname{Im} w \geq 0$, for any $N > 0$, and all $z \in \mathbb{C}^+$, we have the estimate,*

$$\gamma\left(-\frac{v_2(N, z)}{v_1(N, z)}, -\frac{u_2(N, z) + \bar{w}v_2(N, z)}{u_1(N, z) + \bar{w}v_1(N, z)}\right) \leq \frac{1}{\sqrt{I(I+1)}},$$

where $I = I(N, z)$ is the integral defined by $I(N, z) = (\operatorname{Im} z) \int_0^N \operatorname{Im}(u^* H v) dx$.

Proof. Denote the wronskian $W_N(f, g) = f_1(N)g_2(N) - f_2(N)g_1(N)$. Using the Greens's Identity we have,

$$(4.3) \quad \int_0^N v^* H v dx = \frac{1}{2i \operatorname{Im} z} W_N(v, \bar{v}),$$

$$(4.4) \quad \int_0^N \operatorname{Im}(u^* H v) dx = -\frac{1}{2 \operatorname{Im} z} \left(1 - \operatorname{Re} W_N(\bar{u}, v)\right) = \frac{1}{2 \operatorname{Im} z} \left(1 - \operatorname{Re} W_N(u, \bar{v})\right),$$

$$(4.5) \quad |W(u, \bar{v})|^2 = 1 - W(u, \bar{u})W(v, \bar{v}).$$

Now at $x = N$, we have,

$$\gamma^2\left(-\frac{v_2}{v_1}, -\frac{u_2 + \bar{w}v_2}{u_1 + \bar{w}v_1}\right) = -\frac{4}{W(v, \bar{v})W(u + \bar{w}v, \bar{u} + w\bar{v})}.$$

Therefore,

$$\gamma^2 \left(-\frac{v_2}{v_1}, -\frac{u_2 + \bar{w}v_2}{u_1 + \bar{w}v_1} \right) \leq -\frac{4}{W(v, \bar{v})W(u + \bar{w}v, \bar{u} + w\bar{v})}.$$

Let w be real. The denominator on the right side is of the form $A + Bw + Cw^2$, where $A \geq 0, C \geq 0$ and B is real. The denominator has minimum value $A - \frac{B^2}{4C}$. Hence,

$$\begin{aligned} \gamma^2 &\leq \frac{4}{-W(v, \bar{v})W(u, \bar{u}) - \frac{(W(v, \bar{v})(W(u, \bar{v}) - W(\bar{u}, v))^2}{4(-W(v, \bar{v})^2)}} \\ &\leq \frac{-4}{W(v, \bar{v})(W(u, \bar{v}) + \operatorname{Im}(W(u, \bar{v})))^2}. \end{aligned}$$

Using equation 4.1 we get,

$$\begin{aligned} \gamma^2 &\leq -\frac{4}{1 - |W(u, \bar{v})|^2 + (\operatorname{Im}(W(u, \bar{v})))^2} \\ &= \frac{-4}{1 - (\operatorname{Re}W(u, \bar{v}))^2}. \end{aligned}$$

Here,

$$\begin{aligned} 1 - (\operatorname{Re}W(u, \bar{v}))^2 &= (1 - (\operatorname{Re}W(u, \bar{v}))(1 + (\operatorname{Re}W(u, \bar{v}))) \\ &= \left(-2 \operatorname{Im}z \int_0^N \operatorname{Im}(u^* H v) dx \right) \left(1 + 2 \operatorname{Im}z \int_0^N \operatorname{Im}(u^* H v) dx \right). \end{aligned}$$

Therefore,

$$\gamma^2 \leq \frac{1}{I(1+I)} \text{ where } I = \operatorname{Im}z \int_0^N \operatorname{Im}(u^* H v) dx.$$

If w is not real, $w = \operatorname{Re}w + iY, Y > 0$ then $u - iYv$ is also a solution and we have,

$$|W(u - iYv, \bar{v})|^2 = 1 - W(u - iYv, \bar{u} + iY\bar{v})W(v, \bar{v}).$$

Also from above equation,

$$\begin{aligned} \gamma^2 &\leq \frac{-4}{W(v, \bar{v})W(u, \bar{u}) + (\operatorname{Im}(W(u, \bar{v})))^2 + Y^2 W(v, \bar{v})^2 + 2iY \operatorname{Re}W(u, \bar{v})W(v, \bar{v})} \\ &\leq \frac{-4}{W(u - iYv, \bar{u} + iY\bar{v})W(v, \bar{v}) + (\operatorname{Im}W(u - iYv, \bar{v}))^2}. \end{aligned}$$

Since the equation 4.1 is valid for $u - iYv$ we get,

$$\begin{aligned}
\gamma^2 &\leq \frac{-4}{1 - (\operatorname{Re} W(u - iYv, \bar{v}))^2} \\
&= \frac{-4}{(1 + \operatorname{Re} W(u - iYv, \bar{v}))(1 - \operatorname{Re} W(u - iYv, \bar{v}))} \\
&= \frac{-4}{(1 + \operatorname{Re}(W(u, \bar{v}) - iYW(v, \bar{v}))) (1 - \operatorname{Re}(W(u, \bar{v}) - iYW(v, \bar{v})))} \\
&= \frac{-4}{(1 - \operatorname{Re} W(u, \bar{v}) - Y \operatorname{Im} W(v, \bar{v}))(1 + \operatorname{Re} W(u, \bar{v}) + Y \operatorname{Im} W(v, \bar{v}))} \\
&= \frac{-4}{\left(-2 \operatorname{Im} z \int_0^N \operatorname{Im}(u^* H v) dx - \frac{Y}{i} W(v, \bar{v})\right) \left(2 \operatorname{Im} z \int_0^N \operatorname{Im}(u^* H v) dx + 2 + \frac{Y}{i} W(v, \bar{v})\right)} \\
&= \frac{1}{I'(I' + 1)},
\end{aligned}$$

where $I' = \operatorname{Im} z \int_0^N \operatorname{Im}(u^* H v) dx + 2 + \frac{Y}{2i} W(v, \bar{v})$. Notice that $I' \geq I$ since $W(v, \bar{v}) = 2i \operatorname{Im} z \int_0^N v^* H v dx \geq 0$. Hence the lemma is proved for general case. \square

Corollary 4.8. *With the notation above, we have*

$$\lim_{N \rightarrow \infty} \gamma \left(-\frac{v_2(N, z)}{v_1(N, z)}, -\frac{u_2(N, z) + \bar{w}v_2(N, z)}{u_1(N, z) + \bar{w}v_1(N, z)} \right) = 0$$

Proof. From above lemma we have

$$\gamma \left(-\frac{v_2(N, z)}{v_1(N, z)}, -\frac{u_2(N, z) + \bar{w}v_2(N, z)}{u_1(N, z) + \bar{w}v_1(N, z)} \right) \leq \frac{1}{\sqrt{I(I+1)}},$$

where $I = I(N, z)$ is the integral defined by $I(N, z) = (\operatorname{Im} z) \int_0^N \operatorname{Im}(u^* H v) dx$. Want to show that $I \rightarrow \infty$ as $N \rightarrow \infty$. We have,

$$\begin{aligned}
\int_0^N v^* H v dx &= \frac{1}{2i \operatorname{Im} z} W_N(v, \bar{v}) \\
\int_0^N \operatorname{Im}(u^* H v) dx &= -\frac{1}{2i \operatorname{Im} z} (1 - \operatorname{Re} W_N(u, \bar{v})).
\end{aligned}$$

Now lets look at the ratio

$$\begin{aligned}
\frac{2 \operatorname{Im} z \int_0^N \operatorname{Im}(u^* H v) dx + 1}{2i \operatorname{Im} z \int_0^N v^* H v dx} &= \frac{W_N(u, \bar{v}) + W_N(\bar{u}, v)}{2i W_N(v, \bar{v})} \\
&= \frac{W_N(u, \bar{v})}{2i W_N(v, \bar{v})} - \frac{W_N(\bar{u}, v)}{2i W_N(v, \bar{v})} \\
&= \operatorname{Im} C
\end{aligned}$$

where C is the center of the Weyl circle. Since the center of the Weyl circle is continuously depend on z it is uniformly bounded on a compact subset of \mathbb{C}^+ . So

$$\int_0^N \operatorname{Im}(u^* H v) dx + 1 = \operatorname{Im} C \int_0^N v^* H v dx \rightarrow \infty \text{ as } N \rightarrow \infty.$$

This implies that $I \rightarrow \infty$ as $n \rightarrow \infty$. \square

We are now ready to prove Theorem 4.6. We follow the similar approach for the proof of Theorem 4.6 as in [2].

Proof of Theorem 4.6 : Let $A \subset \Sigma_{ac}$, $|A| < \infty$ and let $\epsilon > 0$ be given. We first define a partition $A = A_0 \cup A_1 \cup A_2, \dots \cup A_N$ of disjoint subsets such that $|A_0| < \epsilon$, A_j is bounded for $j \geq 1$. We also require that $m_+(t) \equiv \lim_{y \rightarrow 0+} m_+(t+iy)$ exists and $m_+(t) \in \mathbb{C}^+$ on $\bigcup_{j=1}^N A_j$. To find A_j 's with these properties, first of all put all $t \in A$ for which $m_+(t)$ does not exist or does not lie in \mathbb{C}^+ into A_0 . Then pick (sufficiently large) compact subset $K \subset \mathbb{C}^+$, $K' \subset \mathbb{R}$ so that $A_0 = \{t \in A : m_+(t) \notin K \text{ or } t \notin K'\}$ satisfies $|A_0| < \epsilon$. Subdivide K into finitely many subsets of hyperbolic diameter less than ϵ , then take the inverse images under m_+ of these subsets, and finally intersect with K' to obtain the A_j for $j \geq 1$. It is then true that $m_+(N, t)$ exists and lies in \mathbb{C}^+ for arbitrary $N \in \mathbb{R}$ if $t \in \bigcup_{j=1}^N A_j$. Moreover, we need $m_j \in \mathbb{C}^+$ such that

$$\gamma(m_+(t), m_j) < \epsilon,$$

such m_j can be defined as $m_j = m_+(t_j)$ for any fixed $t_j \in A_j$. By Lemma 4.3, there is a number $y > 0$ such that, for arbitrary Herglotz function F , for any Borel subset S of \mathbb{R} and for all $j = 1, 2, \dots, n$ we have the estimate

$$(4.6) \quad \left| \int_{A_j} \omega_{F(t+iy)}(S) dt - \int_{A_j} \omega_{F(t)}(S) dt \right| \leq \epsilon |A_j|.$$

We can define y for each value of j ; so y is a function of j . However, by taking the minimum value of $y(j)$ as j runs from 1 to n we may assume y is independent of j . Let $M_j(N, z) = \frac{u_2(N, z) + \bar{m}_j v_2(N, z)}{u_1(N, z) + \bar{m}_j v_1(N, z)}$ for any $z \in \mathbb{C}^+$. We shall complete the proof of the theorem by showing that, for $j \geq 1$,

(i): $\int_{A_j} \omega_{m_+(N, t)}(S) dt$ is close to the integral $\int_{A_j} \omega_{\overline{M_j(N, t)}}(S) dt$

where $M_j(N, t) = \frac{u_2(N, t) + \bar{m}_j v_2(N, t)}{u_1(N, t) + \bar{m}_j v_1(N, t)}$ and that

(ii): $\int_A \omega_{m_-(N, t)}(-S) dt$ is close to the same integral for all N sufficiently large.

Proof of (i): We have

$$m_+(N, t) = \frac{u_2(N, t) + m_+(t) v_2(N, t)}{u_1(N, t) + m_+(t) v_1(N, t)}.$$

Hence, for fixed N and t , the mapping from $m_+(t)$ to $m_+(N, t)$ is a Mobius transformation with real coefficients and discriminant $u_1 v_2 - v_1 u_2 = 1$. and γ is invariant under Mobius transformations. Now from 4.1 we see that

$$\gamma\left(m_+(N, t), \frac{u_2(N, t) + m_j v_2(N, t)}{u_1(N, t) + m_j v_1(N, t)}\right) \leq \epsilon \text{ for } j \geq 1 \text{ and } t \in A_j.$$

By equation 4.2 we see that,

$$\left| \omega_{m_+(N, t)}(S) - \omega_{M_j(N, t)}(S) \right| \leq \epsilon,$$

and integration with respect to t over A_j gives the estimate

$$(4.7) \quad \left| \int_{A_j} \omega_{m_+(N, t)}(S) dt - \int_{A_j} \omega_{\overline{M_j(N, t)}}(S) dt \right| \leq \epsilon |A_j|.$$

This holds for all $j = 1, 2, \dots, n$.

Proof of (ii): For $j \geq 1$, define the subset A_j^y of \mathbb{C}^+ , consisting of all $z \in \mathbb{C}^+$ of the

form $z = t + iy$, for $t \in A_j$. Thus A_j^y is the translation of A_j by distance y above the real z -axis. Since A_j is bounded, A_j^y is contained in a compact subset of \mathbb{C}^+ . Hence by Corollary 4.1 there a positive number N_0 such that for $j \geq 1, N \geq N_0$ and $z \in A_j^y$ we have the estimate

$$(4.8) \quad \gamma\left(-\frac{v_2(N, z)}{v_1(N, z)}, -\frac{u_2(N, z) + \bar{m}_j v_2(N, z)}{u_1(N, z) + \bar{m}_j v_1(N, z)}\right) \leq \epsilon.$$

As in the case of y we may choose N_0 to be independent of j . Let $m_-(N, z) = -\frac{v_2(N, z)}{v_1(N, z)}$. Following the similar argument to that in the proof of (i), for any $z = t + iy$ we have the estimate

$$\left| \int_{A_j} \omega_{m_-(N, z)}(-S) dt - \int_{A_j} \omega_{-M_j(N, z)}(-S) dt \right| \leq \epsilon |A_j|,$$

valid for $j \geq 1$ and $N \geq N_0$. Now by Lemma 4.3, equation 4.1 we have,

$$\left| \int_{A_j} \omega_{m_-(N, t)}(-S) dt - \int_{A_j} \omega_{-M_j(N, t)}(-S) dt \right| \leq 3\epsilon |A_j|.$$

Now using the identity $\omega_{-w}(S) = \omega_{\bar{w}}(S)$

$$(4.9) \quad \left| \int_{A_j} \omega_{m_-(N, t)}(-S) dt - \int_{A_j} \omega_{\overline{M_j(N, t)}}(S) dt \right| \leq 3\epsilon |A_j|,$$

which holds for all $j \geq 1$ and $N \geq N_0$ and completes the proof of (ii). Combining the inequalities 4.1 and 4.1 now yields, for $j \geq 1$ and $N \geq N_0$,

$$(4.10) \quad \left| \int_{A_j} \omega_{m_-(N, t)}(-S) dt - \int_{A_j} \omega_{m_+(N, t)}(S) dt \right| \leq 4\epsilon |A_j|.$$

Noting that A_0 was chosen such that $|A_0| \leq \epsilon |A|$ we now have for all $N \geq N_0$,

$$\begin{aligned} & \left| \int_A \omega_{m_-(N, t)}(-S) dt - \int_A \omega_{m_+(N, t)}(S) dt \right| \\ & \leq \sum_{j=0}^n \left| \int_{A_j} \omega_{m_-(N, t)}(-S) dt - \int_{A_j} \omega_{m_+(N, t)}(S) dt \right| \\ & \leq 2|A_0| + 4\epsilon \sum_{j=0}^n |A_j| \leq \epsilon |A| \leq 6\epsilon |A|. \end{aligned}$$

Since ϵ was arbitrary, the theorem follows.

Proof of Theorem 4.1: The proof is basically same as in [15], however let me sketch:

Let $\nu \in \omega(\mu)$. Then there exists a sequence $x_n \rightarrow \infty$ such that $d(S_{x_n}\mu, \nu) \rightarrow 0$. Then by Lemma 4.2 we have that

$$m_{\pm}(x_n, z) \rightarrow M_{\pm}(z) \quad (n \rightarrow \infty),$$

uniformly on compact subset of \mathbb{C}^+ . Here $M_{\pm}(z) = m_{\pm}^{\nu}(0, z)$ are the m functions of the whole line Hamiltonian ν . By Theorem 4.5 we see that

$$m_{\pm}(x_n, z) \rightarrow M_{\pm}(z) \quad (n \rightarrow \infty),$$

in value distribution. That is

$$\lim_{n \rightarrow \infty} \int_A \omega_{m_{\pm}(x_n, t)}(S) dt = \int_A \omega_{M_{\pm}(t)}(S) dt$$

for all Borel sets $A, S \subset \mathbb{R}, |A| < \infty$. Also by Theorem 4.6 we have

$$\int_A \omega_{M_-(t)}(-S) dt = \int_A \omega_{M_+(t)}(S) dt.$$

By Lebesgue differentiation theorem,

$$(4.11) \quad \omega_{M_-(t)}(-S) = \omega_{M_+(t)}(S)$$

for $t \in \Sigma_{ac}$ and all intervals S with rational end points. We can also assume that $M_{\pm}(t) = \lim_{y \rightarrow 0+} M(t + iy)$ exists for these t . Moreover, if $M_-(t) \in \mathbb{R}$, then, by choosing small intervals about this value for $-S$, we see that $M_+(t) = -M_-(t)$. If $M_-(t) \in \mathbb{C}$, then

$$\begin{aligned} \omega_{M_-(t)}(-S) &= \int_{(-S)} \frac{v}{(t-u)^2 + v^2} dt \\ &= - \int_{(S)} \frac{v}{(t+u)^2 + v^2} dt \\ &= \omega_{-\overline{M_-(t)}}(S). \end{aligned}$$

By 4.11 we get,

$$(4.12) \quad M_+(t) = -\overline{M_-(t)}.$$

In the case when $M_-(t) \in \mathbb{R}$ we already have $M_+(t) = -M_-(t)$. So 4.12 holds for almost every $t \in \Sigma_{ac}$, that is $\nu \in \mathcal{R}(\Sigma_{ac})$. This completes the proof.

5. RELATION BETWEEN A SCHRÖDINGER EQUATION / JACOBI EQUATION AND A CANONICAL SYSTEM

5.1. Reduction of Schrodinger Equation to a Canonical System. Let

$$(5.1) \quad -y'' + V(x)y = zy$$

be a Schrodinger equation. Suppose $u(z, z)$ and $v(x, z)$ are the linearly independent solutions of 5.1, satisfying some boundary condition α at 0. Then $u_0 = u(x, 0)$ and $v_0 = v_0(x, 0)$ are the solutions of $-y'' + V(x)y = 0$. Let

$$H(x) = \begin{pmatrix} u_0^2 & u_0 v_0 \\ u_0 v_0 & v_0^2 \end{pmatrix}$$

then the Schrodinger equation 5.1 is equivalent with the canonical system

$$(5.2) \quad Jy' = zHy$$

That is if y solves equation 5.1 then $U(x, z) = T^{-1}(x) \begin{pmatrix} y(x, z) \\ y'(x, z) \end{pmatrix}$ solves the canonical system 5.2.

Alternative Approach : Let

$$(5.3) \quad -y'' + V(x)y = z^2 y$$

be a Schrodinger equation such that $-\frac{d^2}{dx^2} + V(x) \geq 0$ and $y(x, z)$ be its solution. Then $y_0 = y(x, 0)$ be a solution of $-y'' + V(x)y = 0$. Let $W(x) = \frac{y'_0}{y_0}$ then $W^2(x) + W'(x) = V(x)$ so that equation 5.3 becomes

$$(5.4) \quad -y'' + (W^2 + W')y = z^2y.$$

Claim that the equation 5.4 is equivalent with the Dirac system

$$(5.5) \quad Ju' = \begin{pmatrix} z & W \\ W & z \end{pmatrix} u.$$

If y is a solution of 5.4 then $u = \begin{pmatrix} y \\ -\frac{1}{z}(-y' + Wy) \end{pmatrix}$ is a solution of 5.5. Also if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of 5.5 then u_1 is a solution of 5.4. Next we show that the Dirac system 5.5 is equivalent with the Canonical System

$$(5.6) \quad Ju'(x) = zH(x)u(x), \quad H(x) = \begin{pmatrix} e^{2\int_0^x W(t)dt} & 0 \\ 0 & e^{-2\int_0^x W(t)dt} \end{pmatrix}.$$

For if u is a solution of 5.5 then T_0u , where $T_0 = \begin{pmatrix} e^{-\int_0^x W(t)dt} & 0 \\ 0 & e^{\int_0^x W(t)dt} \end{pmatrix}$ is a solution of 5.6.

If we consider a Schrodinger equation of the form,

$$(5.7) \quad -y'' + (W^2 - W')y = z^2y$$

then it is equivalent with the Dirac system

$$(5.8) \quad Ju' = \begin{pmatrix} z & -W \\ -W & z \end{pmatrix} u.$$

In other words, if y is a solution of Schrodinger equation 5.7 then $u = \begin{pmatrix} zy \\ y' + Wy \end{pmatrix}$ is a solution of the Dirac system 5.8. Conversely, if $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ is a solution of the Dirac system 5.8 then u_1 is a solution to the Schrodinger equation 5.7.

The Dirac system 5.8 is equivalent with the canonical system,

$$(5.9) \quad Ju'(x) = zH(x)u(x)$$

where $H(x) = \begin{pmatrix} e^{-2\int_0^x W(t)dt} & 0 \\ 0 & e^{2\int_0^x W(t)dt} \end{pmatrix}$. If u is a solution of the Dirac system 5.8 then $y = T_0u$, $T_0 = \begin{pmatrix} e^{\int_0^x W(t)dt} & 0 \\ 0 & e^{-\int_0^x W(t)dt} \end{pmatrix}$ is a solution of the canonical system 5.9. Conversely if u is a solution of the canonical system 5.9 then $T_0^{-1}u$ is a solution of the Dirac system 5.8.

5.2. Reduction of a Jacobi Equation to a canonical system. Let a Jacobi equation be

$$(5.10) \quad a(n)u(n+1) + a(n-1)u(n) + b(n)u(n) = zu(n).$$

This equation can be written as

$$\begin{pmatrix} u(n) \\ u(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{z-b(n)}{a(n)} \end{pmatrix} \cdot \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix} \\ = [B(n) + zA(n)] \begin{pmatrix} u(n-1) \\ u(n) \end{pmatrix}.$$

Where $B(n) = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{-b(n)}{a(n)} \end{pmatrix}$ and $A(n) = \begin{pmatrix} 0 & 0 \\ 0 & \frac{1}{a(n)} \end{pmatrix}$. Suppose $p(n, z)$ and $q(n, z)$ be the solutions of 5.10 such that $p(0, z) = 1, p(1, z) = 1$ and $q(0, z) = 0, q(1, z) = 1$. So that $p_0(n) = p(n, 0)$ and $q_0(n) = q(n, 0)$ be the solutions of equation 5.10 when $z = 0$. Then

$$\begin{pmatrix} p_0(n) \\ p_0(n+1) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{a(n-1)}{a(n)} & \frac{-b(n)}{a(n)} \end{pmatrix} \begin{pmatrix} p_0(n-1) \\ p_0(n) \end{pmatrix}.$$

(similar expression for $q_0(n)$.) Let $T(n) = \begin{pmatrix} p_0(n-1) & q_0(n-1) \\ p_0(n) & q_0(n) \end{pmatrix}, T(1) = 1$. Then we have the relation $T(n+1) = B(n)T(n)$. Now define

$U(n, z) = T^{-1}(n+1)Y(n, z), Y(n, z) = \begin{pmatrix} p(n-1, z) & q(n-1, z) \\ p(n, z) & q(n, z) \end{pmatrix}$. Then $U(n, z)$ solves an equation of the form

$$(5.11) \quad J(U(n+1, z) - U(n, z)) = zH(n)U(n, z)$$

where $H(n) = JT^{-1}(n+1)A(n)T(n)$. Suppose for each $n \in \mathbb{Z}$, on $(n, n+1)$, H has the form

$$H(x) = h(x)P_\phi, \quad P_\phi = \begin{pmatrix} \cos^2 \phi & \sin \phi \cos \phi \\ \sin \phi \cos \phi & \sin^2 \phi \end{pmatrix}$$

for some $\phi \in [0, \pi)$ and some $h \in L_1(n, n+1), h \geq 0$. (We may choose $h(x) \equiv 1$ on $(n, n+1)$ for each $n \in \mathbb{Z}$) Then the canonical system 1.1 reads

$$u'(x) = -zh(x)JP_\phi u(x).$$

Since the matrices on the right-hand side commute with one another for different values of x , the solution is given by

$$u(x) = \exp\left(-z \int_a^x h(t)dtJP_\phi\right)u(a).$$

However, $P_\phi JP_\phi = 0$, we see that the exponential terminates and we get

$$(5.12) \quad u(x) = \left(1 - z \int_a^x h(t)dtJP_\phi\right)u(a).$$

Clearly equation 5.12 is equivalent with the equation 5.11.

5.3. Relation between Weyl-m functions. We next observe the relation between the Weyl-m functions for Shrodinger equation and the canonical system 1.1.

Lemma 5.1. *For $z \in \mathbb{C}^+$, let $m_s(z), m_c(z)$ denote the Weyl m -functions corresponding to the Schrodinger equation 5.1 and the canonical system 5.2 respectively. Then $m_s(z) = m_c(z)$.*

Proof. Let $T_s(x, z) = \begin{pmatrix} u(x, z) & v(x, z) \\ u'(x, z) & v'(x, z) \end{pmatrix}$ and $T_c(x, z) = \begin{pmatrix} u_1(x, z) & v_1(x, z) \\ u_2(x, z) & v_2(x, z) \end{pmatrix}$ are the transfer matrices corresponding to the Schrodinger equation 5.1 and the canonical system 5.2 respectively. Let $T_0(x) = T_s(x, 0)$ then in 5.2, $H(x) = T_0^* \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0$.

Here $m_s(z)$ is such that $(1, 0)T_s(x, z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} \in L^2(R_+)$ and $m_c(z)$ is such that $T_c(x, z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} \in L^2(H, R_+)$. Note that here, $T_s(x, z) = T_0(x)T_c(x, z)$

It follows that,

$$\begin{aligned} & \int_0^\infty (1, \bar{m}_s)T_s^*(x, z) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_s(x, z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty \\ \Rightarrow & \int_0^\infty (1, \bar{m}_s)T_c^*(x, z)T_0^*(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0(x)T_c(x, z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty \\ \Rightarrow & \int_0^\infty (1, \bar{m}_s)T_c^*(x, z)HT_c(x, z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty. \end{aligned}$$

Since the Weyl-m function $m_c(z)$ is uniquely defined we must have $m_s(z) = m_c(z)$. \square

Lemma 5.2. For $z \in \mathbb{C}^+$, let $m_s(z^2), m_c(z)$ denote the Weyl m -functions corresponding to the Schrodinger equation 5.7 and the canonical system 5.9 respectively. Then $m_s(z^2) = zm_c(z)$.

Proof. Note that, since $H(x) = \begin{pmatrix} e^{2 \int_0^x W(t)dt} & 0 \\ 0 & e^{-2 \int_0^x W(t)dt} \end{pmatrix}$, $f \in L^2(H, \mathbb{R}_+)$ if and only if

$$\int_0^\infty |f_1|^2 e^{2 \int_0^x W(t)dt} dx < \infty, \quad \int_0^\infty |f_2|^2 e^{-2 \int_0^x W(t)dt} dx < \infty.$$

Let $T_s(x, z^2), T_d(x, z)$ and $T_c(x, z)$ denote the transfer matrices of the Schrodinger equation 5.4, the Dirac system 5.5 and the canonical system 5.6 respectively. Then,

$$\begin{aligned} T_s(x, z^2) &= \begin{pmatrix} u(x, z^2) & v(x, z^2) \\ u'(x, z^2) & v'(x, z^2) \end{pmatrix}, \\ T_d(x, z) &= \begin{pmatrix} u(x, z^2) & zv(x, z^2) \\ \frac{u'(x, z^2) - W(x)u(x, z^2)}{z} & v'(x, z) - W(x)v(x, z) \end{pmatrix}, \\ T_c(x, z) &= T_0 T_d(x, z). \end{aligned}$$

It follows that

$$T_d(x, z) = \begin{pmatrix} z & 0 \\ -W & 1 \end{pmatrix} T_s(x, z^2) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix}.$$

So $T_d(x, z) = T_0^{-1}T_c(x, z)$ and

$$T_s(x, z^2) = \frac{1}{z} \begin{pmatrix} 1 & 0 \\ W & z \end{pmatrix} T_d(x, z) \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}.$$

Now we have,

$$\begin{aligned}
& \int_0^\infty (1, \bar{m}_c(z)) T_c^*(x, z) H(x) T_c(x, z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty \\
\Rightarrow & \int_0^\infty (1, \bar{m}_c(z)) T_c^*(x, z) \left[T_0^{-1}(x) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} T_0(x)^{-1} + \right. \\
& \quad \left. T_0(x)^{-1} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_0(x)^{-1} \right] T_c(x, z) \begin{pmatrix} 1 \\ m_s(z) \end{pmatrix} dx < \infty. \\
\Rightarrow & \int_0^\infty (1, \bar{m}_c(z)) T_d^*(x, z) T_0(x) T_0(x)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \\
& \quad T_0(x)^{-1} T_0(x) T_d(x, z) \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty. \\
\Rightarrow & \int_0^\infty (1, \bar{m}_c) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} T_s^*(x, z^2) \begin{pmatrix} \bar{z} & W \\ 0 & 1 \end{pmatrix} \cdot \\
& \quad \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} z & 0 \\ -W & 1 \end{pmatrix} T_s(x, z^2) \begin{pmatrix} \frac{1}{z} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ m_c(z) \end{pmatrix} dx < \infty. \\
\Rightarrow & \int_0^\infty \begin{pmatrix} \frac{1}{z} & \bar{m}_c \end{pmatrix} T_s^*(x, z^2) \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} T_s(x, z^2) \begin{pmatrix} \frac{1}{z} \\ m_c(z) \end{pmatrix} dx < \infty.
\end{aligned}$$

Since the Weyl-m function $m_c(z)$ is uniquely defined we must have

$$m_s(z^2) = z m_c(z).$$

□

Suppose

$$H_+ = \begin{pmatrix} e^{2 \int_0^x W(t) dt} & 0 \\ 0 & e^{-2 \int_0^x W(t) dt} \end{pmatrix}, \quad H_- = \begin{pmatrix} e^{-2 \int_0^x W(t) dt} & 0 \\ 0 & e^{2 \int_0^x W(t) dt} \end{pmatrix}$$

in the canonical system 5.6 and 5.9 respectively. The following lemma shows the relation between their Weyl-m functions.

Lemma 5.3. *If m_{c_+} and m_{c_-} are the Weyl-m function corresponding to the canonical system 5.6 and 5.9 respectively then $m_{c_+} = \frac{-1}{m_{c_-}}$.*

Proof. Notice that $-JH_+J = H_-$. Here u is a solution of $Ju' = zH_+u$ if and only if Ju is a solution of $Ju' = zH_-u$. Let $T_{c_+}(x)$ and $T_{c_-}(x)$ be the transfer matrices and m_{c_+} and m_{c_-} are the Weyl-m functions of the canonical systems with the Hamiltonians H_+ and H_- respectively. Then $T_{c_-}(x) = -JT_{c_+}(x)J$ and

$$\begin{aligned}
& \int_0^\infty (1, \bar{m}_{c_-}) T_{c_-}^*(x) H_- T_{c_-}(x) \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty \\
\Rightarrow & \int_0^\infty (1, \bar{m}_{c_-}) (-JT_{c_+}(x)J)^* H_- (-JT_{c_+}(x)J) \begin{pmatrix} 1 \\ m_{c_-} \end{pmatrix} dx < \infty \\
\Rightarrow & \int_0^\infty \left(1, \frac{-1}{\bar{m}_{c_-}}\right) T_{c_+}^*(x) H_+ T_{c_+}(x) \begin{pmatrix} 1 \\ \frac{-1}{m_{c_+}} \end{pmatrix} dx < \infty.
\end{aligned}$$

Since m_{c_+} is the unique coefficient such that

$$\int_0^\infty (1, \bar{m}_{c_+}) T_{c_+}^*(x) H_+ T_{c_+}(x) \begin{pmatrix} 1 \\ m_{c_+} \end{pmatrix} dx < \infty$$

we have $m_{c+} = \frac{-1}{m_{c-}}$. \square

Theorem 5.4. *Let $\omega(V)$ and $\omega(H)$ are the ω -limit set corresponding to a Schrodinger equation 5.1 and its canonical system 5.2 respectively. Then if $W \in \omega(V)$ then $K \in \omega(H)$ where K is the Hamiltonian corresponding to a canonical system of the Schrodinger equation with W as potential. Conversely, if $K \in \omega(H)$ then K is a Hamiltonian for a canonical system of a Schrodinger equation for some potential $W \in \omega(V)$.*

Proof. Suppose $W \in \omega(V)$ then by definition of ω -limit set there exists a sequence $x_n \rightarrow \infty$ such that $V(x + x_n) \rightarrow W$. Then the corresponding Weyl m-functions also converge, ie $m_s^{V_n}(z) \rightarrow m_s^W(z)$. Let H_n be the Hamiltonian of the canonical system obtained from the Schrodinger equation with the potential $V(x + x_n)$ then $H_n = H(x + x_n)$ then by Lemma 5.1 $m_s^{V_n}(z) = m_c^{H_n}(z)$. and $m_s^W(z) = m_c^H(z)$. Now apply the change of variable by 2.9 and obtain \tilde{H}_n and the corresponding m-function is $m_c^{\tilde{H}_n}(z)$. After the change of variable the corresponding Weyl m-functions are the same up to the change of the point of boundary condition. So the convergence of $m_s^{V_n}(z) = m_c^{H_n}(z)$ implies the convergence of $m_c^{\tilde{H}_n}(z)$. It follows that $m_c^{\tilde{H}_n}(z) \rightarrow m_s^W(z)$. But by Lemma 4.2 $m_s^W(z) = m_c^{\tilde{H}}(z)$ where $m_c^{\tilde{H}}(z)$ is the Weyl m-function for some Hamiltonian \tilde{H} . It follows that $m_c^{\tilde{H}_n}(z) \rightarrow m_c^{\tilde{H}}(z)$. Again by Lemma 4.2 we get $\tilde{H}_n \rightarrow \tilde{H}$ using the change of variable on the canonical system with Hamiltonian \tilde{H} we obtain a Hamiltonian K such that $m_c^{\tilde{H}}(z) = m_c^K(z)$ up to the change of point of boundary condition. It follows that $H_n \rightarrow K$ and so $K \in \omega(H)$. Converse is similar. \square

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